

CALT-68-2370
CITUSC/02-001
hep-th/0201114

TASI lectures: special holonomy in string theory and M-theory

Steven S. Gubser^{1,2}

Lauritsen Laboratory of Physics, 452-48 Caltech, Pasadena, CA 91125

Abstract

A brief, example-oriented introduction is given to special holonomy and its uses in string theory and M-theory. We discuss A_k singularities and their resolution; the construction of a K3 surface by resolving T^4/\mathbf{Z}_2 ; holomorphic cycles, calibrations, and worldsheet instantons; aspects of the low-energy effective action for string compactifications; the significance of the standard embedding of the spin connection in the gauge group for heterotic string compactifications; G_2 holonomy and its relation to $\mathcal{N} = 1$ supersymmetric compactifications of M-theory; certain isolated G_2 singularities and their resolution; the Joyce construction of compact manifolds of G_2 holonomy; the relation of D6-branes to M-theory on special holonomy manifolds; gauge symmetry enhancement from light wrapped M2-branes; and chiral fermions from intersecting branes. These notes are based on lectures given at TASI '01.

January 2002

¹E-mail: ssgubser@theory.caltech.edu

²On leave from Princeton University.

1 Introduction

Special holonomy plays a prominent role in string theory and M-theory primarily because the simplest vacua preserving some fraction of supersymmetry are compactifications on manifolds of special holonomy. The case that has received the most intensive study is Calabi-Yau three-folds (CY_3), first because heterotic string compactifications on such manifolds provided the first semi-realistic models of particle phenomenology, and second because type II strings on Calabi-Yau three-folds exhibit the seemingly miraculous property of “mirror symmetry.” Recently, seven-manifolds with G_2 holonomy have received considerable attention, both because they provide the simplest way to compactify M-theory to four dimensions with $\mathcal{N} = 1$ supersymmetry, and because of some unexpected connections with strongly coupled gauge theory.

The purpose of these two lectures, delivered at TASI '01, is to introduce special holonomy in a way that will make minimal demands on the reader's mathematical erudition,¹ but nevertheless get to the point of appreciating a few deep facts about perturbative and non-perturbative string theory. Some disclaimers are in order: these lectures do not aspire to mathematical rigor, nor to completeness. I have made a perhaps idiosyncratic selection of material that will hopefully serve as a comprehensible invitation to the wider literature. To enhance the appeal of mathematical concepts that may seem abstruse or dreary to the theoretical physicist, I have tried to introduce such concepts either in the context of the simplest possible examples, or in the context of a piece of well-known or important piece of string theory lore. A possible downside of this approach is an occasional loss of clarity.

These lectures were constructed in with the help of some rather standard references: the survey of differential geometry by Eguchi, Gilkey, and Hansen [?]; some of the later chapters of the text by Green, Schwarz, and Witten [?]; appendix B of Polchinski's text [?]; and the original papers by D. Joyce on compact manifolds of G_2 holonomy [?, ?]. The student of string theory wishing to go beyond these lectures will find references [?]-[?] excellent jumping-off points. Also, a set of lectures on special holonomy from a pedagogical but more mathematical point of view has appeared [?].

¹It is my hope that a graduate student who has learned General Relativity, knows the basic facts about Lie groups and their representations, and has at least a nodding acquaintance with string theory, will be able to follow the gist of this presentation. Some of the more advanced topics will require more erudition or background reading.

2 Lecture 1: on Calabi-Yau manifolds

2.1 A_k spaces

The simplest non-trivial Calabi-Yau manifolds are four-dimensional, even though the ones of primary interest in string model building are six-dimensional. To begin our acquaintance with four-dimensional Calabi-Yau's, let's first consider some non-compact orbifolds. In particular, regard four-dimensional flat space as \mathbf{C}^2 (that is, the Cartesian product of the complex plane with itself). There is a natural $SU(2)$ action on \mathbf{C}^2 , where the two complex coordinates form a doublet. Let Γ be a discrete subgroup of $SU(2)$: for example, Γ could be the group of transformations acting on \mathbf{C}^2 like this:

$$\mathbf{Z}_{n+1} : \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \quad (1)$$

where (a, b) are coordinates on \mathbf{C}^2 , and ω is any of the $n+1$ complex numbers satisfying $\omega^{n+1} = 1$. The simplest case would be $n = 1$, so that $\Gamma = \mathbf{Z}_2$, and then the only non-trivial transformation just changes the sign of a and b —that is, it reflects us through the origin of $\mathbf{R}^4 = \mathbf{C}^2$. Now form the orbifold \mathbf{C}^2/Γ . Overlooking the singular point at the origin, this is a manifold of holonomy Γ . More properly, we should call it an orbifold of holonomy Γ .

I haven't even defined holonomy yet, so how can we make such a statement? Consider a two-dimensional analogy: \mathbf{R}^2 admits a natural $SO(2)$ action, and we could also embed $\Gamma = \mathbf{Z}_{n+1} \subset SO(2) = U(1)$ in a natural way. The orbifold \mathbf{R}^2/Γ is a cone of holonomy Γ . This claim we can understand just with pictures, and the complex case is only a slight extension. Suppose, as in figure 1, we take a vector at some point away from the tip of the cone, and parallel translate it around a loop. This is easy to do in the original Cartesian coordinates on \mathbf{R}^2 : the vector doesn't change directions. For the loop that I drew, and for $\Gamma = \mathbf{Z}_6$, the vector comes back to itself rotated by an angle $\varphi = \pi/3$. This is what holonomy is all about: when vectors get parallel-transported around some closed loop, their lengths remain constant but their direction can change, and the holonomy group of an n -dimensional real manifold is the subgroup of $O(n)$ that includes all possible changes of direction for a vector so transported. It is a property of the manifold as a whole, not of any special point or closed loop. So for the example in figure 1, the holonomy group is \mathbf{Z}_6 , acting on the tangent plane of the orbifold in the obvious way. (We can define holonomy in the presence of an orbifold singularity—or any other isolated singularity—just by restricting to paths that avoid the singularity). A generic, smooth, orientable manifold has holonomy $SO(n)$. The smaller the holonomy group, the more special the manifold. If the holonomy group is trivial, the manifold is flat. A non-vanishing Riemann tensor is a local measure of non-vanishing holonomy, but we don't need to know details of this yet.

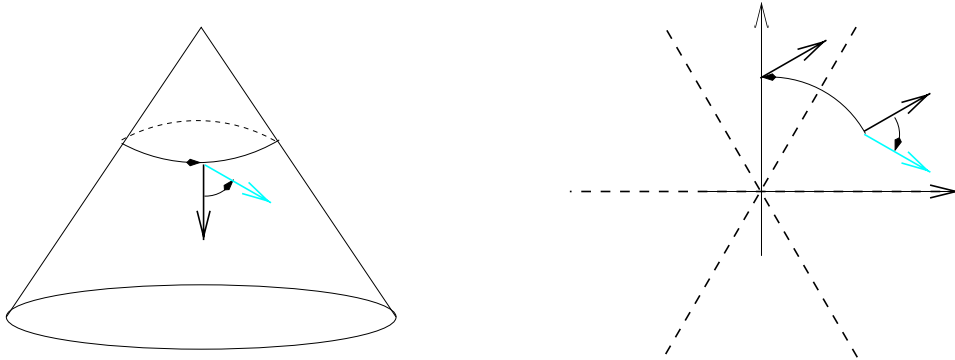


Figure 1: Left: parallel transport of a vector around the tip of a cone changes its direction. Right: the same parallel transport, where the cone is thought of as a plane modded out by a discrete group.

The argument around figure 1 can be repeated to show that $\mathbf{C}^2/\mathbf{Z}_{n+1}$ has holonomy \mathbf{Z}_{n+1} . This orbifold is called an A_n singularity. It's a singular limit of smooth Calabi-Yau manifolds, as we'll see next.

The origin of $\mathbf{C}^2/\mathbf{Z}_{n+1}$ is a curvature singularity. A persistent theme in string theory is the resolution of singularities. Singularity resolution is relatively easy work for Calabi-Yau manifolds because we often have an algebraic description of them. To see how such descriptions arise, note that a and b are double-valued on $\mathbf{C}^2/\mathbf{Z}_2$, but

$$z_1 = a^2, \quad z_2 = b^2, \quad z_3 = ab \quad (2)$$

are single-valued. We can pick any two of these as good local coordinates for $\mathbf{C}^2/\mathbf{Z}_2$. They are related by the equation

$$z_3^2 = z_1 z_2. \quad (3)$$

This is an *equation* for $\mathbf{C}^2/\mathbf{Z}_2$ in \mathbf{C}^3 (and the complex structure is correctly inherited from \mathbf{C}^3 , though the Kahler structure is not—if you don't know what this means, ignore it for now). A *nearby* submanifold of \mathbf{C}^3 , which is completely smooth, is

$$z_3^2 - \epsilon^2 = z_1 z_2, \quad (4)$$

or, after a linear complex change of variables

$$z_1^2 + z_2^2 + z_3^2 = \epsilon^2 \quad (5)$$

where we can, without loss of generality, assume $\epsilon^2 \geq 0$. Clearly, if $\epsilon = 0$, we recover our original $\mathbf{C}^2/\mathbf{Z}_2$ orbifold.

Writing $z_j = x_j + iy_j$, we can recast (5) as

$$\vec{x}^2 - \vec{y}^2 = \epsilon^2, \quad \vec{x} \cdot \vec{y} = 0. \quad (6)$$

Now define $r^2 = \vec{x}^2 + \vec{y}^2 = \sum_{i=1}^3 |z_i|^2$. For large r , our deformed manifold, (4) or (5), asymptotically approaches the original “manifold,” (3). Furthermore, we can easily see that $r^2 \geq \epsilon^2$, and that for $r^2 = \epsilon^2$, we have to have $\vec{y} = 0$ and $\vec{x}^2 = \epsilon^2$: this is a sphere of radius ϵ . In fact, the manifold defined by (5) can also be described as the cotangent bundle over S^2 , denoted T^*S^2 . To understand this, parametrize S^2 using a real vector \vec{w} with $\vec{w}^2 = \epsilon^2$. Any 1-form on S^2 can be expressed as $\vec{y} \cdot d\vec{w}$, where $\vec{y} \cdot \vec{w} = 0$. The space of all possible 1-forms over a point on S^2 is \mathbf{R}^2 . The total space of 1-forms over S^2 , which we have called T^*S^2 , is thus some fibration of \mathbf{R}^2 over S^2 . And we’ve just learned that this total space is parametrized by (\vec{w}, \vec{y}) with $\vec{w}^2 = \epsilon^2$ and $\vec{y} \cdot \vec{w} = 0$. Now if we change variables from \vec{w} to $\vec{x} = \vec{w}\sqrt{1 + \vec{y}^2/\epsilon^2}$, we reproduce (6).

Let’s review what’s happened so far. The original orbifold, $\mathbf{C}^2/\mathbf{Z}_2$, is a cone over S^3/\mathbf{Z}_2 . Note that S^3/\mathbf{Z}_2 is smooth, because the \mathbf{Z}_2 action on S^3 induced from (1) has no fixed points. (It’s the identification of antipodal points). In fact, S^3/\mathbf{Z}_2 is the $SO(3)$ group manifold. The higher S^3/\mathbf{Z}_{n+1} are also smooth because the \mathbf{Z}_{n+1} action has no fixed point on the $U(1)$ Hopf fiber. Our algebraic resolution of the singularity led us to a smooth manifold which was asymptotic to the cone over S^3/\mathbf{Z}_2 , but had a S^2 of radius ϵ at its “tip” rather than a singularity. This is illustrated schematically in figure 2.

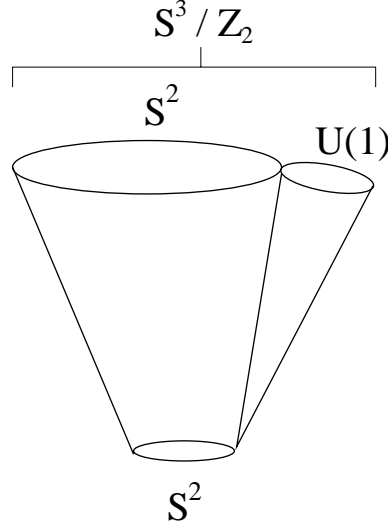


Figure 2: S^3/\mathbf{Z}_2 is a $U(1)$ fibration over S^2 , and in the interior, the $U(1)$ shrinks but the S^2 doesn’t.

This was just the beginning, because we have yet to really specify the metric on the manifolds specified by (5). We should *not* simply suppose that the metric naturally inherited from \mathbf{C}^3 is the one we want. In fact, the beautiful truth for these manifolds is that there is a one-parameter family of Ricci-flat Kahler metrics respecting the obvious $SO(3)$ symmetry of the equation (5) (explanation of the word “Kahler” will

be forthcoming). These metrics have $SU(2)$ holonomy. This means, precisely, that the spin connection, $\omega_\mu{}^a{}_b$, generically an $SO(4)$ gauge field, lies entirely in one $SU(2)$ subgroup of $SO(4) = SU(2)_L \times SU(2)_R$. By convention we could say that the holonomy group is $SU(2)_L$. Then a *constant* right-handed spinor field ϵ_R obviously satisfies

$$\nabla_\mu \epsilon = \partial_\mu \epsilon_R + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \epsilon_R = 0, \quad (7)$$

just because the second term is a linear combination of the generators of rotation in $SU(2)_L$, under which ϵ_R is invariant. The integrability condition of the equation (7) is

$$[\nabla_\mu, \nabla_\nu] \epsilon_R = \frac{1}{4} R_{\mu\nu ab} \gamma^{ab} \epsilon_R = 0, \quad (8)$$

for any ϵ_R such that $\gamma_5 \epsilon_R = \epsilon_R$. (That's an equivalent way of saying that a spinor is right-handed). Thus, for *any* spinor ϵ (right-handed or not),

$$\begin{aligned} R_{\mu\nu ab} \gamma^{ab} (1 - \gamma_5) \epsilon &= R_{\mu\nu ab} \gamma^{ab} (1 - \gamma^1 \gamma^2 \gamma^3 \gamma^4) \epsilon \\ &= R_{\mu\nu ab} \left(\gamma^{ab} - \frac{1}{2} \epsilon^{abcd} \gamma_{cd} \right) \epsilon \\ &= \left(R_{\mu\nu ab} - \frac{1}{2} \epsilon_{abcd} R_{\mu\nu}{}^{cd} \right) \epsilon = 0, \end{aligned} \quad (9)$$

and, evidently, this can be true if and only if the Riemann tensor is self-dual:

$$R_{\mu\nu ab} = \frac{1}{2} \epsilon_{abcd} R_{\mu\nu}{}^{ab}. \quad (10)$$

Because (10) looks a lot like the equations for an instanton in non-abelian gauge theory, the metric of $SU(2)$ holonomy on (5) is known as a “gravitational instanton.” This metric is known explicitly, and is called the Eguchi Hansen space, or EH_2 :

$$ds^2 = \frac{dr^2}{1 - (\epsilon/r)^4} + r^2 \left(\sigma_x^2 + \sigma_y^2 + (1 - (\epsilon/r)^4) \sigma_z^2 \right), \quad (11)$$

where

$$\begin{aligned} \sigma_x &= \cos \psi d\theta + \sin \psi \sin \theta d\phi & \sigma_y &= -\sin \psi d\theta + \cos \psi \sin \theta d\phi \\ \sigma_z &= d\psi + \cos \theta d\phi. \end{aligned} \quad (12)$$

It's worth noting that the metric on S^3 can be written as

$$ds_{S^3}^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 = d\theta^2 + \sin^2 \theta d\phi^2 + (d\psi + \cos \theta d\phi)^2, \quad (13)$$

and the 1-forms σ_i are invariant under the left action of $SU(2)$ on $S^3 = SU(2)$. To cover S^3 once, we should let $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$, and $0 \leq \psi < 4\pi$. On the other hand, in the expression (11) for the Eguchi-Hansen metric, ψ is restricted to range

over $[0, 2\pi)$. Thus the metric for large r is indeed a cone over $SO(3) = S^3/\mathbf{Z}_2$: the \mathbf{Z}_2 action on S^3 is just $\psi \rightarrow \psi + 2\pi$.

Clearly, (11) is the promised one-parameter family of metrics on the resolved A_1 singularity. The parameter is ϵ , and one can verify that the S^2 at $r = \epsilon$ indeed has radius ϵ in the metric (11). The $SO(3)$ symmetry of (5) is included in the $SU(2)$ invariance of the σ_i .

Having thoroughly disposed of this simplest example of a special holonomy metric, it's worth saying that a Calabi-Yau n -fold is, in general, a manifold of $2n$ real dimensions whose holonomy group is $SU(n)$ (or a subgroup thereof—but usually we mean that the holonomy group is precisely $SU(n)$). Any particle physicist will have encountered the embedding of $SU(2)$ in $SO(4)$ as one of the “chiral” subgroups. The inclusion of $SU(n)$ in $SO(2n)$ can be described by saying that the $2n$ real-dimensional vector representation of $SO(2n)$, which we could write as $(x_1, y_1, x_2, y_2, \dots, x_n, y_n)$, becomes the n -dimensional complex representation of $SU(n)$, which we could write as (z_1, z_2, \dots, z_n) where $z_j = x_j + iy_j$. Having a holonomy group $SU(n)$ necessarily means that the Calabi-Yau n -fold is Ricci-flat: this is a frequently observed property of special holonomy manifolds. But not always: for instance, Kahler manifolds are $2n$ real-dimensional manifolds with holonomy group $U(n)$ (or a subgroup thereof), and these aren't Ricci-flat unless the holonomy group is contained in $SU(n)$.

The results described so far for the A_1 singularity admit interesting generalizations in several directions:

- A_n singularity: Here the natural, single-valued coordinates are $z_1 = a^{n+1}$, $z_2 = b^{n+1}$, and $z_3 = ab$, and they are related by the equation $z_3^{n+1} = z_1 z_2$, which can be deformed to $\prod_{k=1}^{n+1} (z_3 - \xi_k) = z_1 z_2$. If the constants ξ_k are all distinct, the deformed equation defines a smooth manifold in \mathbf{C}^3 . All such manifolds admit Ricci-flat metrics. The “tip of the resolved cone” is a rather more complicated geometry now: there are $n(n+1)/2$ holomorphic embeddings of S^2 into a resolved A_n singularity, but only n are distinct in homology. Thus $b_2 = n$ for these manifolds.
- D_n and E_6, E_7, E_8 are the other finite subgroups of $SU(2)$. One can find algebraic descriptions and resolutions of \mathbf{C}^2/Γ for these cases as well, in a manner similar to the A_n cases.
- Another important class of $SU(2)$ holonomy metrics is the multi-center Taub-NUT solutions. They are $U(1)$ fibrations over \mathbf{R}^3 , with metric

$$ds_{TN}^2 = H d\vec{r}^2 + H^{-1} (dx^{11} + \vec{C} \cdot d\vec{r})^2 \quad \text{where} \quad (14)$$

$$\nabla \times \vec{C} = -\nabla H, \quad H = \epsilon + \frac{1}{2} \sum_{i=1}^{n+1} \frac{R}{|\vec{r} - \vec{r}_i|}.$$

Clearly, H is a harmonic function on \mathbf{R}^3 . There appears to be a singularity in (14) when \vec{r} is equal to one of the \vec{r}_i , but in fact the manifold is completely smooth, for all \vec{r}_i distinct, provided x^{11} is made periodic with period $2\pi R$. When $k > 1$ of the \vec{r}_i coincide, there is an A_{k-1} singularity. An efficient way to see this is that, with k of the \vec{r}_i coincident, we've made the "wrong" choice of the periodization of x^{11} : the right choice of period, to make the local geometry non-singular, would have been $2\pi kR$. We can get from the right choice to the wrong choice by modding out x^{11} by \mathbf{Z}_k , and now what's left is to convince yourself that this is the same \mathbf{Z}_k action that produced A_{k-1} from \mathbf{C}^2 . If $\epsilon > 0$, the geometry far for large r is metrically the product $S^1 \times \mathbf{R}^3$ (see figure 3). If $\epsilon > 0$, asymptotically the space is a cone over S^3/\mathbf{Z}_n : that is, in (14) with $\epsilon = 0$ we have exhibited explicitly the general metric of $SU(2)$ holonomy on a resolved A_n singularity.

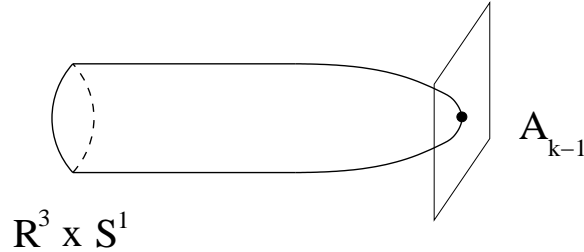


Figure 3: Single-center Taub-NUT ($k = 1$ in (14)) interpolates between $\mathbf{R}^3 \times S^1$ and an \mathbf{R}^4 which is well-approximated by the tangent plane to the tip of the cigar. Having k centers coincident amounts to orbifolding by \mathbf{Z}_k in the S^1 direction, and results in an A_{k-1} at the tip of the cigar.

- It's now possible to outline the construction of a compact Calabi-Yau 2-fold, also known as a K3 surface. It's worth remarking that all compact, smooth Calabi-Yau 2-folds with precisely $SU(2)$ holonomy are homeomorphic (not at all an obvious result). Suppose we start with $T^4/\mathbf{R}^4/\mathbf{Z}^4$, where the lattice \mathbf{Z}^4 is just the one generated by the unit vectors $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$. Now let us identify by the action of \mathbf{Z}_2 which reflects through the origin: this is precisely the \mathbf{Z}_2 action that we used to define the A_1 singularity, so evidently there will be such a singularity at the origin. Actually, on T^4 as a whole, there are 16 fixed points of the \mathbf{Z}_2 action, and each is an A_1 singularity: they are at points (r_1, r_2, r_3, r_4) , where each r_i can be chosen independently as 0 or $1/2$. It's worth verifying that these are all the fixed points. A good way to go about it is to show that the fixed points in \mathbf{R}^4 of the combined action of \mathbf{Z}^4 and \mathbf{Z}_2 are the images of the 16 points we just mentioned under action of the \mathbf{Z}^4 . A look at figure 4a) may help. At any rate, we now have a compact but singular space, and its holonomy is obviously \mathbf{Z}_2 , with the usual caveat of

avoiding fixed points (the argument is the same as always: translate a vector around the space, and the most it can do is switch its sign). The “Kummer construction” of a smooth K3 space is to cut out a region of radius R around each of the 16 A_1 singularities, and replace it by a copy of the Eguchi-Hansen space, cut off at the same finite radius R , and having an S^2 of radius $\epsilon > 0$ at its tip. This procedure works topologically because the surface $r = R$ of an Eguchi-Hansen space is S^3/\mathbf{Z}_2 , and that’s the same space as we got by cutting out a region around the A_1 singularity: the boundary of B^4/\mathbf{Z}_2 , where B^4 is a ball with boundary S^3 . See figure 4b). The metric does not quite match after we’ve pasted in copies of EH_2 , but it nearly matches: the errors are $O(\epsilon^4/R^4)$. Neglecting these small errors, we have a smooth manifold of $SU(2)$ holonomy: the crucial point here is that each EH_2 has the *same* $SU(2)$ subgroup of $SO(4)$ as its holonomy group, namely the $SU(2)$ which contains the original discrete \mathbf{Z}_2 holonomy of the A_1 singularity—and that \mathbf{Z}_2 is the same for all 16 fixed points. A *non-trivial* mathematical analysis shows that the $O(\epsilon^4/R^4)$ can be smoothed out without enlarging the holonomy group. It’s easy to understand from this analysis that K3 has 22 homologically distinct 2-cycles: T^4 started out with 6 that are undisturbed by the \mathbf{Z}_2 orbifolding (think of their cohomological partners, for instance $dr_1 \wedge dr_2$, obviously \mathbf{Z}_2 even); and each EH_2 adds one to the total because of the unshrunk S^2 at its tip. As remarked earlier, all K3 surfaces are homeomorphic. Hence all of them have second Betti number $b_2 = 22$.

It’s worth reflecting for a moment on why we were able to get so far in the study of the A_k spaces just by manipulating complex equations like $z_3^2 = z_1 z_2$. This defining equation for the A_1 space does not determine its metric, but it does determine its complex structure. That is, the notion of holomorphicity is inherited from \mathbf{C}^3 to the subspace defined by the algebraic equation. Another way to say it is we automatically have a distinguished way of assembling four real coordinates into two complex coordinates. Note that we haven’t said anything yet about the metric! The natural notion of a metric that is “compatible” with a given complex structure is what’s called a Kahler metric: it is one which can be expressed locally as

$$ds^2 = 2g_{i\bar{j}} dz^i d\bar{z}^{\bar{j}} \quad \text{where} \quad g_{i\bar{j}} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^{\bar{j}}}, \quad (15)$$

for some function $K(z^i, \bar{z}^{\bar{j}})$ which is called the Kahler potential. Evidently, $K(z^i, \bar{z}^{\bar{j}})$ can be modified by the addition of a holomorphic or an anti-holomorphic function. It is quite straightforward to show that the Christoffel connection associated with a Kahler metric preserves the splitting of the tangent plane into holomorphic and anti-holomorphic pieces: for instance, if a vector points in the z_1 direction, then after parallel transport, it may have components in the z_1 and z_2 directions, but none in the

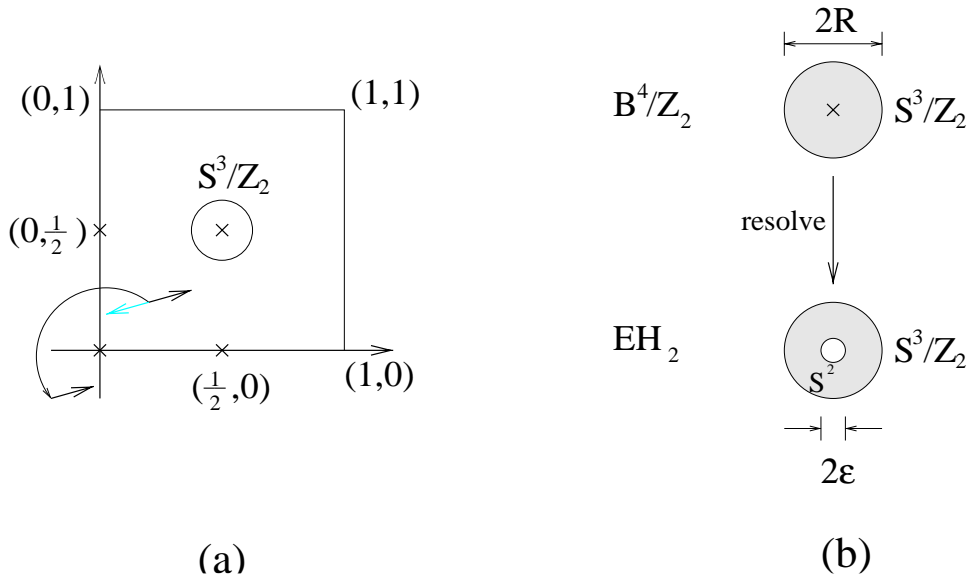


Figure 4: (a): Schematic description of T^4/\mathbf{Z}_2 . The unit cell of a square torus is quotiented by the action of a \mathbf{Z}_2 whose fixed points are indicated by x's. (Actually there would be $2^4 = 16$ such fixed points for T^4 , but we could only draw T^2 here). Each fixed point is an A_1 singularity, so the boundary of a region around it is S^3/\mathbf{Z}_2 in the quotient space. The quotient is an orbifold of \mathbf{Z}_2 holonomy: parallel transport of a vector along a curve, plus its reflected image, are shown. (b) We resolve a B^4/\mathbf{Z}_2 region around each A_1 singularity into the central portion of an Eguchi-Hansen space, with an unshrunk S^2 of radius ϵ .

\bar{z}^1 and \bar{z}^2 directions. This is why Kahler metrics on an n -complex-dimensional space necessarily have holonomy $U(n)$.

Yau proved that if a smooth, compact manifold, admitting a complex structure and a Kahler metric, obeys a certain topological condition (vanishing of the first Chern class), then it's possible to find a Ricci-flat Kahler metric. (Some further facts are not so hard to show: the Ricci-flat metric is unique given the cohomology class of the Kahler form; and Ricci-flat Kahler metrics are precisely those with holonomy contained in $SU(n)$). By virtue of Yau's theorem, we can go far in the study of $SU(n)$ holonomy manifolds just by manipulating simple algebraic equations: the equations specify a topology and a complex structure (inherited from the complex structure in the flat space or projective space in which we write the defining equations) and provided we can demonstrate the (rather weak) topological hypotheses of Yau's theorem, we can be sure of the existence of a $SU(n)$ holonomy metric even if we can't write it down. Perhaps the simplest way to look at it is that you get to $U(N)$ holonomy just by knowing the complex structure. The Kahler metric is detailed and difficult information, but a lot of interesting facts can be learned without knowing much about it other than its existence.

We have discussed some of the simplest special holonomy manifolds, and sketched the Kummer construction for a compact K3; but much much more remains unsaid. There are highly developed ways of constructing Calabi-Yau three-folds, of which elliptic fibration, toric geometry, and the intersection of algebraic varieties in complex projective spaces deserve special mention. Far too much is in the literature to even summarize here; but the interested reader will find much already in the references to these lectures.

2.2 Non-linear sigma models and applications to string theory

I find it irresistible at this point to detour into some applications of notions from special holonomy to supersymmetry and string theory. In four dimensions, the most general renormalizable lagrangian for a single chiral superfield, $\Phi = \phi + \theta^\alpha \psi_\alpha + \theta^\alpha \theta_\alpha F$, is

$$\mathcal{L} = \int d^4\theta \Phi^\dagger \Phi + \left(\int d^2\theta W(\Phi) + h.c. \right), \quad (16)$$

with $W(\Phi)$ some cubic polynomial. Let us work in Euclidean signature. The most general *effective* action for several chiral superfields (that is, a totally general local form up to two derivative) is the following:

$$\begin{aligned} \mathcal{L} &= \int d^4\theta K(\Phi_i, \Phi_i^\dagger) + \left(\int d^2\theta W(\Phi) + h.c. \right) \\ &= g_{i\bar{j}} \partial_\mu \phi^i \partial^\mu \bar{\phi}^{\bar{j}} + g_{i\bar{j}} \psi^i \not{\partial} \bar{\psi}^{\bar{j}} + g^{i\bar{j}} \frac{\partial W}{\partial \phi^i} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{j}}} + \left(\frac{\partial^2 W}{\partial \phi^i \partial \phi^j} \psi^i \psi^j + h.c. \right) + \dots, \end{aligned} \quad (17)$$

where in the last line I have eliminated the auxiliary fields F_i through their algebraic equations of motion. In expanding things out in components I have left out various interaction terms, and I have not been particularly careful with all factors of 2 and signs.

If the superpotential is 0, then the lagrangian is just

$$L = g_{i\bar{j}}(\phi^k, \bar{\phi}^{\bar{k}}) \partial_\mu \phi^i \partial^\mu \bar{\phi}^{\bar{j}} + \text{fermions}, \quad (18)$$

which is just a non-linear sigma model with a Kahler target. There are various reasons to be interested in the lagrangians (17) and (18), but let us point out one that is particularly relevant to string theory. If we make a dimensional reduction to two dimensions, setting $\phi^i = Z^i / \sqrt{2\pi\alpha'}$, then we obtain an action

$$S = \frac{1}{2\pi\alpha'} \int d^2z g_{i\bar{j}}(Z^k, \bar{Z}^{\bar{k}}) \left(\partial_z Z^i \partial_{\bar{z}} \bar{Z}^{\bar{j}} + \partial_{\bar{z}} Z^i \partial_z \bar{Z}^{\bar{j}} + \text{fermions} \right). \quad (19)$$

The bosonic part written out explicitly is precisely the so-called Polyakov action, $S_{\text{Pol}} = \frac{1}{2\pi\alpha'} \int d^2z g_{ab} \partial_z X^a \partial_{\bar{z}} X^b$, written in terms of complex variables, $Z^j \propto X^{2j-1} + iX^{2j}$.

The action (19) describes strings propagating on a Kahler manifold. We know (see for instance E. D'Hoker's lectures at this school) that conformal invariance forces this manifold to be ten-dimensional and *Ricci flat*, in the leading approximation where α' is small compared to characteristic sizes of the manifold. For instance, the target space could be a Calabi-Yau manifold times flat space: this is part of the standard strategy for getting four-dimensional models out of the heterotic string (more on this later).

A simpler example would be for the target space just to be \mathbf{R}^6 times the Eguchi-Hansen space, EH_2 . (In fact, we could even use the singular orbifold limit, provided $\int_{S^2} B_2 = \pi$; but it is too much to consider hear in detail how string physics can be smooth on a singular geometry). Pursuing our simple $\mathbf{R}^6 \times EH_2$ example a little further: an obvious thing for a string to do is to wrap the S^2 in EH_2 . The string is then an instanton with respect to the \mathbf{R}^6 directions, and to compute its contribution to the path integral, the first thing we have to know is the minimal classical action for such a string.² To this end, it is worth recalling that the Polyakov action coincides with the Nambu-Goto action after the worldsheet metric is eliminated through its algebraic equation of motion. So the minimal action will be attained by a worldsheet wrapped on the minimal area S^2 . Finding this S^2 is straightforward work, since we have the explicit metric for EH_2 : it's obviously $r = \epsilon$. But for a more general discussion, it's worth introducing a little more technology, in the form of the Kahler form

$$J = ig_{i\bar{j}}(Z^k, Z^{\bar{k}})dZ^i \wedge d\bar{Z}^{\bar{j}}. \quad (20)$$

Since both $g_{i\bar{j}}\partial_z Z^i \partial_{\bar{z}} \bar{Z}^{\bar{j}}$ and $g_{i\bar{j}}\partial_{\bar{z}} Z^i \partial_z \bar{Z}^{\bar{j}}$ are everywhere positive quantities, it's clear that

$$\frac{1}{2\pi\alpha'} \int_{S^2} J = \frac{1}{2\pi\alpha'} \int_{S^2} d^2z g_{i\bar{j}} \left(\partial_z Z^i \partial_{\bar{z}} \bar{Z}^{\bar{j}} - \partial_{\bar{z}} Z^i \partial_z \bar{Z}^{\bar{j}} \right) \leq S_{\text{Pol}} \quad (21)$$

with equality precisely if $g_{i\bar{j}}\partial_z Z^i \partial_{\bar{z}} \bar{Z}^{\bar{j}} = 0$, which is equivalent to $\partial_{\bar{z}} Z^i = 0$ for all i . This last equation expresses the condition that the map $z \rightarrow Z^i(z)$ is a *holomorphic embedding* of the worldsheet into the target spacetime. Obviously, we could consider anti-holomorphic embeddings, and prove in an analogous way that precisely they saturate the inequality $\frac{1}{2\pi\alpha'} \int_{S^2} J \leq -S_{\text{Pol}}$. A string anti-holomorphically embedded in EH_2 would just be one at $r = \epsilon$, wrapping the S^2 with the opposite orientation. Thus we have world-sheet instantons and world-sheet anti-instantons.

The inequality (21) is deceptively simple. Actually it illustrates a very powerful notion: calibration. To see things in a properly general light, first note that we didn't need the two-cycle to be S^2 : it could have been any homologically non-trivial two-cycle,

²We would eventually have in mind formulating a string theory in $\mathbf{R}^{5,1}$ via Wick rotation from \mathbf{R}^6 —or, in the more physically interesting case of a Calabi-Yau three-fold, in $\mathbf{R}^{3,1}$ via Wick rotation from \mathbf{R}^4 —but we carry on in the hallowed tradition of doing all computations in Euclidean signature until the very end.

call it Σ . Furthermore, we could have derived a pointwise form of the inequality in (21) (obvious since we didn't need any integrations by parts to get the inequality we did derive). That pointwise form would say that the pullback of the Kahler form J to the worldsheet is equal to a multiple of the volume form (defined through the induced metric on the worldsheet), and the multiple is a function that never exceeds 1. A final important ingredient to the setup of a calibration is that J is closed, $dJ = 0$. This arises because $J = i\partial\bar{\partial}K$, where ∂ is the exterior derivative with respect to the Z^i 's, and $\bar{\partial}$ is the exterior derivative with respect to the $\bar{Z}^{\bar{i}}$'s. So to state the whole setup once and for all and with full generality: a calibration is a closed p -form which restricts (or, more precisely, pulls back) onto any p -submanifold to a scalar multiple of the induced volume form, where the multiple is nowhere greater than 1; and a calibrated cycle is one whose induced volume form precisely coincides with the pullback of the calibration form. An inequality like (21) then ensures that the volume of the calibrated cycle is minimal among all possible cycles in its homology class: this is because the integral of the calibrating form (i.e. the left hand side of (21)) depends only on the homology of what you're integrating it over.

Suppose now we have a compactification of string theory from ten dimensions to four on a (compact) Calabi-Yau three-fold, CY_3 . If we pick a basis N^A of homology two-cycles for CY_3 , then we could define the Kahler parameters as $v^A = \int_{N^A} J$. From the preceding discussion, v^A is just the minimal area two-cycle in a given equivalence class. A natural complexification of v^A is

$$T^A = \int_{N^A} (J + iB), \quad (22)$$

where B is the NS 2-form, assumed to have $dB = 0$. The T^A are the so-called complexified Kahler moduli of the Calabi-Yau compactification. The claim is that they become massless complex fields in four dimensions. To see this in precise detail, we should perform a rigorous Kaluza-Klein reduction. Without going that far, we can convince ourselves of the claim by expanding

$$J + iB = \sum_A T^A \omega_A, \quad (23)$$

where the ω_A are harmonic two-forms with $\int_{N^A} \omega_B = \delta_B^A$; and (23) is basically the beginnings of a Kaluza-Klein reduction, where T^A depends only on the four non-compact dimensions. Since the left hand side of (23) is harmonic (or may at least be made so by a gauge choice) and the ω_A are harmonic, the T^A are indeed massless fields in four dimensions. Compactification on CY_3 preserves 1/4 of supersymmetry (a theme to be developed more systematically in the next lecture), which means $\mathcal{N} = 1$ supersymmetry in $d = 4$ for a heterotic string compactification, and $\mathcal{N} = 2$ supersymmetry in $d = 4$ for a type II string compactification. Since we have at least $\mathcal{N} = 1$ supersymmetry,

the complex scalar fields T^A must be components of chiral superfields, with an action of the form (16), for some Kahler target manifold that describes all possible values of the complexified Kahler moduli for a given Calabi-Yau compactification.³ What a mouthful! Now comes the nice part: having learned that the T^A are massless fields based on an argument that applied for *any* Calabi-Yau, we can confidently say that $V = 0$ identically, so also the superpotential $W = 0$. These are classical statements, because the argument that the T^A were massless was based on classical field equations. However, as is often the case, W is protected against contributions from loops by the unbroken $\mathcal{N} = 1$ supersymmetry. More precisely, a Peccei-Quinn symmetry for $\int_{N^A} B$, plus holomorphy, protects W against *all* perturbative string corrections. There are in fact non-perturbative corrections that come from the world-sheet instantons discussed above: the action of such an instanton is

$$S = \frac{1}{2\pi\alpha'} \int_{N^A} (J + iB) = \frac{T^A}{2\pi\alpha'}, \quad (24)$$

and because of the explicit T^A -dependence, we obviously must expect some nonperturbative $e^{-T^A/2\pi\alpha'}$ contribution to W to arise from these instantons.

This is about all one can learn about the dynamics of complexified Kahler moduli for CY_3 compactifications of superstrings based on $\mathcal{N} = 1$ supersymmetry. It's actually quite a lot: we have non-linear sigma model dynamics on a Kahler manifold whose complex dimension is b_2 of the CY_3 , corrected only non-perturbatively in the small dimensionless parameters $T^A/2\pi\alpha'$. More can be learned, however, if there is $\mathcal{N} = 2$ supersymmetry—that is, for CY_3 compactifications of a type II superstring. Then one can show that the Kahler metric on the moduli space follows from the Kahler potential

$$K = -\log \mathcal{W}(\text{Re } T^A) \quad \mathcal{W} = \int_{CY_3} J \wedge J \wedge J, \quad (25)$$

where J is the Kahler form of the CY_3 (but K is the Kahler potential for the many-dimensional moduli space, and as such is a function of T^A and $\bar{T}^{\bar{A}}$). Explaining how (25) arises from $\mathcal{N} = 2$ supersymmetry would take us too far afield; it is enough for us to know that, whereas $\mathcal{N} = 1$ supersymmetry usually protects only the holomorphic object W from corrections, $\mathcal{N} = 2$ supersymmetry tightly constrains the Kahler form as well, protecting it in this case from all perturbative string corrections. There are worldsheet instanton corrections, as before.

A substantial omission in our treatment is that we haven't discussed complex structure moduli. Understanding them, and also the worldsheet origin of both types of moduli, is crucial to the formulation of mirror symmetry in string theory. The reader

³We have not substantially constrained how these moduli might couple to other sorts of matter. This issue is beyond the scope of the present lectures.

may wish to consult TASI lectures from previous years (for instance [?]) for an introduction to these fascinating topics.

A truly remarkable property of heterotic string theory dynamics is that the form (25) continues to hold true, modulo similar non-perturbative corrections, in $\mathcal{N} = 1$ compactifications of the heterotic string with the “standard embedding” of the spin connection in the gauge group. “Standard embedding” means that one sets gauge potentials $A_\mu^I{}_J$ in a particular $SU(3)$ subgroup of $SO(32)$, or of $E_8 \times E_8$, equal to the spin connection $\omega_{\mu b}^a$ of the CY_3 . In contrast to the results presented so far, the fact that (25) persists for these $\mathcal{N} = 1$ constructions goes beyond anything one could understand based only on low-energy effective field theory, and is truly stringy in its origin. Before returning to the narrower venue of special holonomy, let us then detour into a demonstration of this claim. Amusingly, almost all the tools we will use have already been introduced.

The basic point is that, for the standard embedding, the CY_3 part of the heterotic worldsheet CFT is identical to the corresponding part of the type II worldsheet CFT. Because the heterotic CFT factorizes into a $\mathbf{R}^{3,1}$ part, a CY_3 part (to be described), and an “extra junk” part, the physical dynamics of the CY_3 is the same for the heterotic and type II constructions. It’s as if there were a “secret” $\mathcal{N} = 2$ supersymmetry in the heterotic string. To write down type II superstring propagation on a CY_3 , we need to make the non-linear sigma model (19) explicitly supersymmetric. With the help of superfields

$$\mathcal{X}^a = X^a + i\theta\psi^a + i\bar{\theta}\tilde{\psi}^a + \theta\bar{\theta}F^a, \quad (26)$$

one can write a simple supercovariant worldsheet action:

$$\begin{aligned} S_{CY_3} &= \frac{1}{2\pi\alpha'} \int d^2z d^2\theta g_{ab}(\mathcal{X}) D_{\bar{\theta}}\mathcal{X}^a D_{\theta}\mathcal{X}^b \\ &= \frac{1}{2\pi\alpha'} \int d^2z \left[g_{ab}(X) \partial X^a \bar{\partial} X^b + g_{ab}(\psi^a D_{\bar{z}}\psi^b + \tilde{\psi}^a D_z\tilde{\psi}^b) \right], \end{aligned} \quad (27)$$

where the second line obtains after auxiliary fields have been algebraically eliminated. The covariant derivatives are defined as follows:

$$\begin{aligned} D_{\theta} &= \partial_{\theta} + \theta\partial_z & D_{\bar{\theta}} &= \partial_{\bar{\theta}} + \bar{\theta}\partial_z \\ D_{\bar{z}}\psi^a &= \partial_{\bar{z}}\psi^a + \partial_{\bar{z}}X^b\Gamma_{bc}^a(X)\psi^c \\ D_z\tilde{\psi}^a &= \partial_z\tilde{\psi}^a + \partial_zX^b\Gamma_{bc}^a(X)\tilde{\psi}^c. \end{aligned} \quad (28)$$

The complicated second term in $D_{\bar{z}}\psi^a$ and $D_z\tilde{\psi}^a$ are the pull-backs of the Calabi-Yau connection to the string worldsheet. The full action for type II superstrings on $\mathbf{R}^{3,1} \times CY_3$ is

$$S_{II} = \frac{1}{2\pi\alpha'} \int d^2z d^2\theta \eta_{\mu\nu} D_{\bar{\theta}}\mathcal{X}^{\mu} D_{\theta}\mathcal{X}^{\nu} + S_{CY_3}. \quad (29)$$

The heterotic string possesses only the anti-holomorphic fermions $\tilde{\psi}^M$: instead of the corresponding ten holomorphic fermions ψ^M , the heterotic string has 32 holomorphic fermions λ^I . (The choice of GSO projection determines whether we have $SO(32)$ or $E_8 \times E_8$ as the gauge group. In the latter case, $SO(16) \times SO(16)$ is manifest in the above description, as rotations of the λ^I 's in two sets of 16. For further details about the heterotic string, standard string theory texts should be consulted). The action of the heterotic string is

$$S_{Het} = \frac{1}{2\pi\alpha'} \int d^2z \left[g_{MN} \partial X^M \bar{\partial} X^N + g_{MN} \tilde{\psi}^M D_z \tilde{\psi}^N + \delta_{IJ} \lambda^I \mathcal{D}_{\bar{z}} \lambda^J + \frac{1}{2} F_{MN}^{IJ} \lambda^I \lambda^J \tilde{\psi}^M \tilde{\psi}^N \right] \quad (30)$$

where the only new derivative we need to define is

$$\mathcal{D}_{\bar{z}} \lambda^I = \partial_{\bar{z}} \lambda^I + A_M^{IJ}(X) \partial_{\bar{z}} X^M \lambda^J, \quad (31)$$

the second term being the heterotic gauge field pulled back to the worldsheet. (It's easiest to think of the A_M^{IJ} either as $SO(32)$ gauge fields, or in the $E_8 \times E_8$ case as $SO(16) \times SO(16)$ gauge fields, which have to be augmented by some other fields to make up the full $E_8 \times E_8$, but these other fields will never be turned on in our construction). Now for the punch-line: we can embed S_{CY_3} into S_{Het} by “borrowing” six of the λ^I to replace the six lost ψ^a . More explicitly,

$$\begin{aligned} \psi^I &\equiv e_a^I(X) \psi^a \rightarrow \lambda^I, & \omega_a^{IJ} &\rightarrow A_a^{IJ}, & R_{ab}^{IJ} &\rightarrow F_{ab}^{IJ} \\ \delta_{IJ} e_a^I e_b^J &= g_{ab} & I &= 1, \dots, 6 & D_{\bar{z}} \psi^I &= \partial_{\bar{z}} \psi^I + \partial_{\bar{z}} X^a \omega_a^{IJ}(X) \psi^J. \end{aligned} \quad (32)$$

Thus, quite literally, we are embedding a particular $SU(3) \subset SU(4) = SO(6) \subset SO(16)$, and the $SO(16)$ is either part of $SO(32)$ or E_8 . Clearly, $SU(3) \subset SU(4)$ in only one way, and $SO(6) \subset SO(16)$ so that $SO(6)$ rotates only 6 components of the real vector representation of $SO(16)$.

A lesson to remember, even if not all the details registered, is that the spin connection can be thought of as just another connection (acting on the tangent bundle so that $\nabla_a v^I = \partial_a v^I + \omega_a^{IJ} v^J$), and it is not only well-defined, but in fact quite convenient, to set some of the gauge fields of the heterotic string equal to the spin connection of $SU(3)$ holonomy that we know exists on any Calabi-Yau. Less minimal choices have been extensively explored—see for example D. Waldram's lectures at this school [?].

3 Lecture 2: on G_2 holonomy manifolds

Given that all string theories can be thought of as deriving from a single eleven-dimensional theory, M-theory, by a chain of dualities, it is natural to ask what are the sorts of seven-dimensional manifolds we can compactify M-theory on to obtain

minimal supersymmetry in four-dimensions.⁴ This is the most obvious string theory motivation for studying seven-manifolds of G_2 holonomy, as indeed we shall see that M-theory on such manifolds leads to $\mathcal{N} = 1$ supersymmetry in $d = 4$.

But what is G_2 ? It can be defined as the subgroup of $SO(7)$ whose action on \mathbf{R}^7 preserves the form

$$\begin{aligned}\varphi &= dy^1 \wedge dy^2 \wedge dy^3 + dy^1 \wedge dy^4 \wedge dy^5 + dy^1 \wedge dy^6 \wedge dy^7 + dy^2 \wedge dy^4 \wedge dy^6 \\ &\quad - dy^2 \wedge dy^5 \wedge dy^7 - dy^3 \wedge dy^4 \wedge dy^7 - dy^3 \wedge dy^5 \wedge dy^6 \\ &\equiv \frac{1}{6} \varphi_{abc} dy^a dy^b dy^c.\end{aligned}\tag{33}$$

The φ_{abc} happen to be the structure constants for the imaginary octonions. We will not use this fact, but instead take the above as our *definition* of G_2 . Let's now do a little group theory. $SO(7)$ has rank 3 and dimension 21. Three obvious representations are the vector **7**, the spinor **8**, and the adjoint **21**. G_2 , on the other hand, has rank 2 and dimension 14. See figure 5.

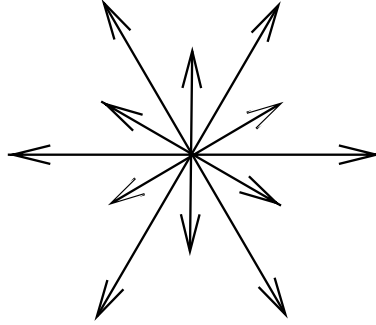


Figure 5: The Dynkin diagram for G_2 . The weights comprising the **7** are the six short roots plus one node at the origin.

It has two obvious representations: the fundamental **7** (comprising the short roots plus one weight at the origin) and the adjoint **14**. As a historical note, it's worth mentioning that G_2 enjoyed brief popularity as a possible group to describe flavor physics: the **7** was supposed to be the multiplet of pseudoscalar mesons. That looked OK until it was realized that the η had to be included in this multiplet, which made the **8** of $SU(3)$ clearly superior. Besides, spin $3/2$ baryons almost filled out the **10** of $SU(3)$, and then the discovery of the Ω^- completed that multiplet and clinched

⁴Some people might prefer the phrasing, “All string theories are special limits of a mysterious theory, M-theory, of which another limit is eleven-dimensional supergravity.” I will prefer to use M-theory in its more restrictive sense as a theory emphatically tied to eleven dimensions—in other words, the as-yet unknown quantum completion of eleven-dimensional supergravity.

$SU(3)$'s victory. To return to basic group theory, it's worth noting some branching rules:

$$\begin{array}{ll}
SO(7) \supset G_2 & G_2 \supset SU(3) \\
\mathbf{21} = \mathbf{14} \oplus \mathbf{7} & \mathbf{7} \supset \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{1} \\
\mathbf{7} = \mathbf{7} & \mathbf{14} = \mathbf{8}_{\text{adj}} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \\
\mathbf{8} = \mathbf{7} \oplus \mathbf{1} &
\end{array} \tag{34}$$

The second rule in the right column suggests another construction of G_2 , as $SU(3)$ plus generators in the $\mathbf{3}$ and the $\bar{\mathbf{3}}$ —this is similar to the construction of E_8 from $SO(16)$ plus spinor generators.

The construction of G_2 as a subgroup of $SO(7)$ makes it clear that G_2 is a possible holonomy group of seven-manifolds. Before explaining this in detail, let us re-orient the reader on the concept of holonomy. Recall that on a generic seven-manifold, parallel transport of a vector around a closed curve brings it back not to itself, but to the image of itself under an $SO(7)$ transformation which depends on the curve one chooses. See figure 6. The reason that the transformation is in $SO(7)$ is that the *length* of the

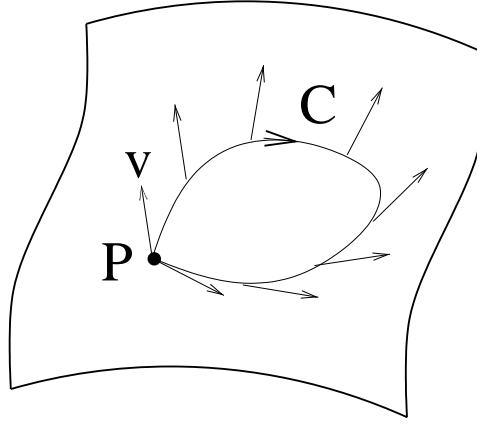


Figure 6: Parallel transport of a vector v around a curve C . Upon returning to the point of origin P , v has undergone some rotation, which for a seven-manifold is an element of $SO(7)$.

vector is preserved: parallel transport means $t^\mu \nabla_\mu v^\alpha = 0$ along the curve C , and this implies $t^\mu \nabla_\mu (g_{\alpha\beta} v^\alpha v^\beta) = 0$ (because $\nabla_\mu g_{\alpha\beta} = 0$); so indeed the length of the vector v is the same, all the way around the curve. Suppose we now choose some seven-bein e_μ^a , satisfying $\delta_{ab} e_\alpha^a e_\beta^b = g_{\alpha\beta}$. Parallel transporting all seven of these 1-forms around our closed curve C results in

$$e_\alpha^a \rightarrow O^a_b e_\mu^b, \tag{35}$$

where $O^a_b \in SO(7)$. Parallel transport in this context means transport with respect to the covariant derivative $\nabla_\nu e_\mu^a = \partial_\nu e_\mu^a - \Gamma_{\nu\mu}^\rho e_\rho^a$: that is, we treat a merely as a label.

One often defines another covariant derivative, D_μ , such that a flat index a results in an extra term involving the spin connection: thus for instance

$$D_\nu e_\mu^a = \partial_\nu e_\mu^a - \Gamma_{\nu\mu}^\rho e_\rho^a + \omega_n u^a_b e_\mu^b. \quad (36)$$

The spin connection can then be defined by the equation $D_\nu e_\mu^a = 0$.

Thus far our setup has nothing to do with G_2 : we have merely explained (or re-explained) some standard aspects of differential geometry. Now suppose our seven-manifold is special, in that for *some* choice of seven-bein e_μ^a , the three-form

$$\begin{aligned} \varphi &= e^1 \wedge e^2 \wedge e^3 + e^1 \wedge e^4 \wedge e^5 + e^1 \wedge e^6 \wedge e^7 + e^2 \wedge e^4 \wedge e^6 \\ &\quad - e^2 \wedge e^5 \wedge e^7 - e^3 \wedge e^4 \wedge e^7 - e^3 \wedge e^5 \wedge e^6 \\ &\equiv \frac{1}{6} \varphi_{abc} e^a e^b e^c \end{aligned} \quad (37)$$

satisfies $\nabla_\mu \varphi_{\alpha\beta\gamma} = 0$. That means, in particular, that if we parallel transport φ around C , it comes back to itself. Rephrasing this statement using (35) and the concise form $\varphi = \frac{1}{6} \varphi_{abc} e^a e^b e^c$, we see that $\varphi_{abc} O^a_d O^b_e O^c_f = \varphi_{def}$. So the $SO(7)$ transformation O^a_b is actually an element of G_2 ; and since the curve C was arbitrary, the manifold's holonomy group is G_2 .

The presentation of the previous paragraph is in the order that my intuition suggests; however it's actually backwards according to a certain logic. A mathematician might prefer to state it this way: it so happens that preservation of the form (33) under a general linear transformation *implies* preservation of the metric δ_{ab} . So we could start with a manifold M_7 endowed only with differential structure, choose a globally defined three-form φ on it, *determine* the metric $g_{\mu\nu}$ in terms of φ ,⁵ determine the connection ∇_μ in terms of $g_{\mu\nu}$, and then ask that $\nabla_\mu \varphi_{\alpha\beta\gamma} = 0$ in order to have a G_2 holonomy manifold. This amounts to a set of hugely non-linear differential equations for the three-form coefficients $\varphi_{\alpha\beta\gamma}$.

The decomposition $\mathbf{8} = \mathbf{7} \oplus \mathbf{1}$ of the spinor of $SO(7)$ into representations of G_2 is important, because it means that G_2 holonomy manifolds admit precisely one covariantly constant spinor. To construct it, start at any point P , choose ϵ at P as the singlet spinor according to the above decomposition, and then parallel transport ϵ everywhere over the manifold. There is no path ambiguity because the spinor always stays in the singlet representation of G_2 . All other spinors are shuffled around by the holonomy: only the one we have constructed satisfies $\nabla_\mu \epsilon = 0$. The equation for preserved supersymmetry in eleven-dimensional supergravity, with the four-form $G_{(4)}$ set to zero, is

$$\delta\psi_\mu = \nabla_\mu \eta = 0. \quad (38)$$

⁵A formula for the metric in terms of φ will be given in section 3.1. The validity of this formula already requires that φ have some non-degeneracy properties. A more careful analysis can be found in [?].

For an eleven-dimensional geometry $\mathbf{R}^{3,1} \times M_7$, where M_7 has G_2 holonomy, the solutions for η in (38) are precisely ϵ tensored with a spinor in $\mathbf{R}^{3,1}$: that is, compactification on M_7 preserves one eighth of the possible supersymmetry, which amounts to $\mathcal{N} = 1$ in $d = 4$. It can also be shown that if a manifold has precisely one covariantly constant spinor ϵ , then its holonomy group is G_2 , or at least a large subgroup thereof. One can in fact construct the covariant three-form φ as a bilinear in ϵ .

It would seem that G_2 holonomy compactifications of 11-dimensional supergravity would be of utmost phenomenological interest; however, one should recall Witten's proof [?] that compactifications of 11-dimensional supergravity on any *smooth* seven-manifold cannot lead to chiral matter in four dimensions. With a modern perspective, we conclude that we should therefore be studying *singularities* in G_2 holonomy manifolds, or branes, or some other defects where chiral fermions might live.

The usual starting point for investigating singularities in an n -dimensional manifold is to look at non-compact manifolds which are asymptotically conical:

$$ds_n^2 \sim dr^2 + r^2 d\Omega_{n-1}^2 \quad (39)$$

for large r . Note that if \sim were replaced by an exact equality, then the metric ds_n^2 would be singular at $r = 0$ unless $d\Omega_{n-1}^2$ is the metric on a unit $(n-1)$ -sphere. In the previous lecture, we encountered a prime example of this sort of singularity: A_k singularities in four-manifolds locally have the form (39) with $d\Omega_3^2$ being the metric of the Lens space S^3/\mathbf{Z}_{k+1} . Another frequently discussed example is the conifold singularity in Calabi-Yau three-folds: this is locally a cone over the coset space $T^{11} = SU(2) \times SU(2)/U(1)_{\text{diag}}$. The conifold admits a Calabi-Yau metric that is known explicitly, as are certain resolutions of the singularity which remain Calabi-Yau (much like the resolutions of the A_k singularities discussed in the previous lecture). As remarked previously, one can gain tremendous insight into Calabi-Yau singularities through algebraic equations: for instance, the A_1 space and the conifold can be described, respectively, via the equations $\sum_{i=1}^3 z_i^2 = 0$ and $\sum_{i=1}^4 z_i^2 = 0$. Sadly, there is no such algebraic tool known for describing singular or nearly singular G_2 holonomy manifolds. And in fact, there are essentially only three known asymptotically conical metrics of G_2 holonomy. The bases of the cones are \mathbf{CP}^3 , $\frac{SU(3)}{U(1) \times U(1)}$, and $S^3 \times S^3$, but the metrics $d\Omega_6^2$ that appear through (39) in the G_2 holonomy metrics are not the obvious metrics on these spaces (just as, in fact, the metric on T^{11} induced by the Calabi-Yau metric on the conifold is not quite the metric suggested by the coset structure). The three metrics admit isometry groups $SO(5)$, $SU(3)$, and $SU(2)^3$, respectively. (Don't get confused between isometry and holonomy: isometry means that after some transformation the metric is the same as before, whereas holonomy tells us how complicated the transformation properties of vectors are under parallel transport). And at the "tip" of the three respective asymptotically conical metrics, an S^4 , or a

\mathbf{CP}^2 , or an S^3 , remains finite. See figure 7 for a schematic depiction of the S^4 case.

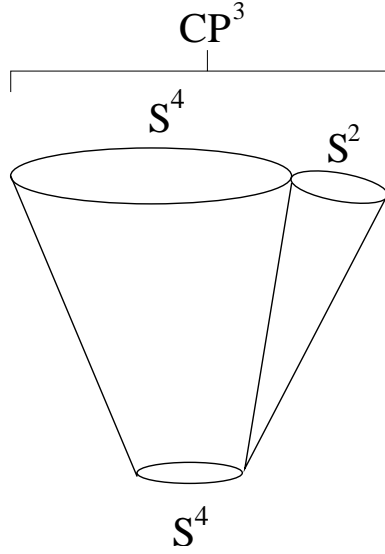


Figure 7: \mathbf{CP}^3 a S^2 fibration over S^4 , and in the interior, the S^2 shrinks but the S^4 doesn't.

We may describe the explicit G_2 holonomy metrics in terms only slightly more complicated than the explicit metric (11) for EH_2 . For the $SO(5)$ symmetric case, one has

$$ds_7^2 = \frac{dr^2}{1-r^{-4}} + \frac{1}{4}r^2(1-r^{-4})(d\mu^i + \epsilon^{ijk}A^j\mu^k)^2 + \frac{1}{2}r^2ds_4^2, \quad (40)$$

where ds_4^2 is the $SO(5)$ symmetric metric on a unit S^4 , the μ^i are three Cartesian coordinates on S^2 , subject to $\sum_{i=1}^3(\mu^i)^2 = 1$, and A_μ^i is an $SU(2)$ gauge field on S^4 carrying unit instanton number. We can be a little more explicit about this gauge field, as follows. S^4 is a space of $SO(4)$ holonomy, but $SO(4) \approx SU(2)_L \times SU(2)_R$, and the spin connection $\omega_{\mu ab}$ is decomposable into $SU(2)_L$ and $SU(2)_R$ pieces as $\omega_{\mu ab}^{L,R} = \omega_{\mu cd} \left(\delta_a^c \delta_b^d \pm \frac{1}{2} \epsilon^{cd}_{ab} \right)$. The gauge field A_μ^i can be taken proportional to $\sigma_{ab}^i \omega_{\mu ab}^L$, where σ_{ab}^i are the Pauli matrices. The \mathbf{CP}^2 case is identical to the above discussion, only one takes ds_4^2 to be the $SU(3)$ symmetric metric on a \mathbf{CP}^2 whose size is such that the Ricci curvature is three times the metric (as is true for a unit S^4). Clearly, when $r = 1$, the S^2 part of the metric shrinks to nothing, while the S^4 or \mathbf{CP}^2 remains finite. Topologically, the whole space is a bundling of \mathbf{R}^3 over S^4 or \mathbf{CP}^2 , and the base is the corresponding S^2 bundle over S^4 or \mathbf{CP}^2 .

The $SU(2)^3$ symmetric case is actually more “elementary,” in the sense that we do not need to discuss gauge fields. The metric is

$$ds_7^2 = \frac{dr^2}{1-r^{-3}} + \frac{1}{9}r^2(1-r^{-3})(\nu_1^2 + \nu_2^2 + \nu_3^2) + \frac{r^2}{12}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \quad (41)$$

where $\nu_i = \Sigma_i - \frac{1}{2}\sigma_i$, and Σ_i and σ_i are left-invariant one-forms on two different S^3 's. Clearly, only one of these S^3 's stays finite as $r \rightarrow 1$. Topologically, the whole space is a bundling of \mathbf{R}^4 over S^3 . Any G_2 holonomy metric can be rigidly rescaled without changing its holonomy group: thus we can claim to have exhibited three one-parameter families of G_2 holonomy metrics, each parametrized by the S^4 , or \mathbf{CP}^2 , or S^3 , that remains unshrunk. A perfectly conical metric has an isometry under scaling: $dr^2 + r^2 d\Omega_6^2 \rightarrow d(\Omega r)^2 + (\Omega r)^2 d\Omega_6^2$ for any constant factor Ω . Thus the asymptotics of the rescaled metrics is always the same. And the limit where the unshrunk space at the center goes to zero volume is an exactly conical metric. Considerably more detail on these G_2 holonomy metrics can be found in the original papers [?, ?].

It may seem that our discussion of G_2 holonomy metrics is remarkably unenlightening and difficult to generalize. This is true! Despite more than 15 years since the discovery of the metrics (40) and (41), there are few generalizations of them, and little else known about explicit G_2 holonomy metrics. One interesting generalization of (41) is the discovery of less symmetric versions where, as with Taub-NUT space, there is a $U(1)$ fiber which remains finite at infinity [?]. Nevertheless, there are several generally useful observations to make at this point:

- G_2 holonomy implies Ricci flatness. A mathematically rigorous proof is straightforward, but a nice physical argument is that G_2 holonomy on M_7 implies unbroken supersymmetry for eleven-dimensional supergravity on $\mathbf{R}^{3,1} \times M_7$ with $G_{(4)} = 0$; and supersymmetry implies the equations of motion, which for $G_{(4)} = 0$ are precisely Ricci flatness. Ricci flatness is a common feature of special holonomy manifolds: $SU(n)$ and $Spin(7)$ holonomy manifolds are also necessarily Ricci-flat; but $U(n)$ holonomy manifolds are not.
- The condition $\nabla_\mu \varphi_{\alpha\beta\gamma} = 0$ can be shown to be *equivalent* to the apparently *weaker* condition $d\varphi = 0 = d*\varphi$. These first order equations can be considerably easier to solve than $R_{\mu\nu} = 0$. The three-forms for each of the three “classical” G_2 holonomy metrics are explicitly known, but we would not gain much from exhibiting their explicit forms.
- The three-form φ , as well as its Hodge dual $*\varphi$, are calibrations. Examples of calibrated three- and four-cycles are the unshrunk S^3 , S^4 , and \mathbf{CP}^2 at $r = 1$ in the metrics (40) and (41). An M2-brane on the unshrunk S^3 would be a supersymmetric instanton in M-theory, similar to the worldsheet instantons arising from strings on holomorphic curves. An exploration of such instantons (including their zero modes) can be found in [?].
- M-theory has a 3-form potential, C . Just as we formed $J + iB$ in string theory, so we can form $\varphi + iC$, and then $\int_{S^3}(\varphi + iC)$ is the analog of a complexified Kahler

parameter $\int_{NA}(J + iB)$. As an example of the use of this analogy, one may show that $M2 - brane$ instantons make a contribution to the superpotential whose dominant behavior is $\exp\{-\tau_{M2} \int_{S^3}(\varphi + iC)\}$. Perturbative corrections to the classical superpotential are forbidden by the usual holomorphy plus Peccei-Quinn symmetry argument. However, in contrast to the case of type II superstrings, or heterotic strings with the standard embedding, where the Kahler potential could be related to a holomorphic prepotential, it is difficult to say anything systematic about the Kahler potential for G_2 compactifications: no “hidden” supersymmetry is available, and perturbative corrections at all orders seem to be allowed.

Recall that after discussing the A_1 singularity in detail, we were able to go on to construct a smooth, compact $SU(2)$ holonomy manifold by resolving A_1 singularities of an orbifold of T^4 by a discrete subgroup of $SU(2)$. Around each fixed point, we cut out little regions of the orbifold, whose local geometry was B^4/\mathbf{Z}_2 (B^4 being a unit ball), and we replaced them by cut-off copies of the Eguchi-Hansen space EH_2 . Smoothing out the small discontinuities in the metric at the joining points, without losing $SU(2)$ holonomy, was an interesting subtlety that we left for the mathematical literature. It turns out that a very similar strategy suffices to construct smooth compact G_2 holonomy manifolds. This is called the Joyce construction, and it was the way in which the first explicit examples of compact G_2 holonomy manifolds were found [?, ?]. We start with an orbifold T^7/Γ , where T^7 is the square unit torus parametrized by $\vec{x} = (x^1, \dots, x^7)$, and Γ is a discrete subgroup of G_2 , to be specified below. Γ has a set of fixed points S which, in the upstairs picture, is locally a three-dimensional submanifold of T^7 . Each fixed point is an A_1 singularity. The key step is to replace $S \times B^4/\mathbf{Z}_2$ by $S \times EH_2$, and then argue that after smoothing out the small discontinuities, the resulting smooth manifold has G_2 holonomy.

A particular example of this strategy begins with the discrete subgroup Γ generated by the following three transformations:

$$\begin{aligned}\alpha : \vec{x} &\rightarrow (-x^1, -x^2, -x^3, -x^4, x^5, x^6, x^7) \\ \beta : \vec{x} &\rightarrow (-x^1, \tfrac{1}{2} - x^2, x^3, x^4, -x^5, -x^6, x^7) \\ \gamma : \vec{x} &\rightarrow (\tfrac{1}{2} - x^1, x^2, \tfrac{1}{2} - x^3, x^4, -x^5, x^6, -x^7).\end{aligned}\tag{42}$$

These generators have several nice properties which make the Joyce construction work:

- They commute. The group Γ is \mathbf{Z}_2^3 .
- They preserve a three-form φ of the form (33) (with an appropriate relabellings of the x^i 's as y^j 's), so indeed the action of Γ induced by (42) on the tangent space of T^7 is a subgroup of the usual action of $G_2 \subset SO(7)$. (This is what we mean, precisely, by $\Gamma \subset G_2$).

- The generators α , β , and γ each individually has a fixed point set in T^7 consisting of 16 T^3 's. β and γ act freely on the fixed point set of α , and similarly for the fixed point sets of β and γ .
- The 48 T^3 's coming from the fixed point sets of the generators α , β , and γ are disjoint, but the 16 from α are permuted by β and γ , and similarly for the 16 from β and from γ . Thus on the quotient space, S consists of 12 disjoint T^3 's.

Since S has 12 disjoint components in the quotient space, we must ensure when replacing $S \times B^4/\mathbf{Z}_2$ by $S \times EH_2$ that all 12 $SU(2)$'s are in the *same* G_2 . To this end, we exploit the fact that EH_2 is hyperkahler, which is to say its metric is Kahler with respect to three different complex structures. In practice, what this means is that there exist covariantly constant ω_1 , ω_2 , and ω_3 (the three possible Kahler forms), which in local coordinates at any given point can be written as

$$\begin{aligned}\omega_1 &= dy^1 \wedge dy^4 + dy^2 \wedge dy^3 \\ \omega_2 &= dy^1 \wedge dy^3 - dy^2 \wedge dy^4 \\ \omega_3 &= dy^1 \wedge dy^2 + dy^3 \wedge dy^4.\end{aligned}\tag{43}$$

(On the singular space $\mathbf{C}^2/\mathbf{Z}_2$, the y^i could be taken as real coordinates for \mathbf{C}^2). Now, the cotangent space of any one of the 12 T^3 's is spanned by three one-forms: dx^i , dx^j , and dx^k for some choice of i , j , and k . For a correct ordering of i , j , and k , and correct identification of the y coordinates in (43) with the remaining four x coordinates, the form

$$\varphi = \omega_1 \wedge dx^1 + \omega_2 \wedge dx^2 + \omega_3 \wedge dx^3 + dx^1 \wedge dx^2 \wedge dx^3\tag{44}$$

is precisely the original three-form (33), written at the location of each T^3 in a way which the replacement $B^4/\mathbf{Z}_2 \rightarrow EH_2$ clearly preserves. This is the reasoning that allows us to say that the holonomy group is still contained in G_2 after the resolution. As before, we gloss over the subtlety of smoothing out the discontinuities; this is well treated in Joyce's original papers [?, ?]. There it is also shown that the moduli space of G_2 metrics is locally $H^3(M_7, \mathbf{R}) = \mathbf{R}^{43}$ for this example. The moduli space of M-theory on this manifold is locally $H^3(M_7, \mathbf{C})$ because of the complexification $\varphi + iC$. The Kahler potential on this moduli space is probably *hard* to compute beyond the classical level, for the reasons explained above.

Beautiful and impressive though the Joyce construction is, we still seem as yet rather stuck in mathematics land in our study of G_2 holonomy manifolds. There are two main themes in the relation of G_2 holonomy to string theory. One, which we will not discuss, centers on a relationship with strongly coupled gauge theories, developed in [?, ?, ?]. The other, perhaps more obvious relation, is with configurations of D6-branes in type IIA string theory. To begin, we should recall the basic ansatz relating type IIA string

theory to M-theory:

$$ds_{11}^2 = e^{-2\Phi/3} ds_{str}^2 + e^{4\Phi/3} (dx^{11} + C_\nu dx^\nu)^2, \quad (45)$$

where C_μ is the Ramond-Ramond one-form of type IIA, and Φ is the dilaton. The classical geometry for $n+1$ flat, parallel D6-branes can be cast in the form (45):

$$\begin{aligned} ds_{11}^2 &= ds_{\mathbf{R}^{6,1}}^2 + H d\vec{r}^2 + H^{-1} (dx^{11} + \vec{\omega} \cdot d\vec{r})^2 \\ \nabla \times \vec{\omega} &= -\nabla H \quad e^\Phi = H^{-3/4} \quad H = 1 + \frac{1}{2} \sum_{i=1}^{n+1} \frac{R}{|\vec{r} - \vec{r}_i|} \\ ds_{str}^2 &= H^{-1/2} ds_{\mathbf{R}^{6,1}}^2 + H^{1/2} d\vec{r}^2 \quad R = g_{str} \sqrt{\alpha'}. \end{aligned} \quad (46)$$

Here \vec{r} parametrizes the three directions perpendicular to the D6-branes, whose centers are at the various \vec{r}_i . Since the eleven-dimensional geometry is the direct product of flat $\mathbf{R}^{6,1}$ and multi-center Taub-NUT, the holonomy group is $SU(2)$, and hence 1/2 of supersymmetry is preserved. It is indeed appropriate, since parallel D6-branes preserve this much supersymmetry. A more general observation is that, since D6-branes act as sources only for the metric, the Ramond-Ramond one-form, and the dilaton, and these fields are organized precisely into the eleven-dimensional metric, *any* configuration of D6-branes that solves the equations of motion must lift to a Ricci-flat manifold in eleven dimensions; and if the configuration of D6-branes is supersymmetric, then the eleven-dimensional geometry must have at least one covariantly constant spinor, and hence special holonomy. In particular, if there is a factor of flat $\mathbf{R}^{3,1}$ in the geometry, and some supersymmetry is unbroken the rest of it must be a seven-manifold whose holonomy is contained in G_2 . In the example above, the seven-manifold is \mathbf{R}^3 times multi-center Taub-NUT.

Before developing this theme further, it seems worthwhile to explore the dynamics of $n+1$ parallel D6-branes, as described in (46), a little further. Recall that we learned in lecture 1 that there are n homologically non-trivial cycles for the $n+1$ -center Taub-NUT geometry: topologically, this is identical to a resolved A_n singularity. Thus there exist n harmonic, normalizable two-forms, call them ω^i . These forms are localized near the centers of the Taub-NUT space, and they are the cohomological forms dual to the n non-trivial homology cycles. Furthermore, there is one additional normalizable 2-form on the Taub-NUT geometry, which can be constructed explicitly for $n=0$, but which owes its existence to no particular topological property. Let us call this form ω^0 . If we expand the Ramond-Ramond three-form of type IIA as

$$C_{(3)} = \sum_{i=0}^n \omega^i \wedge A_i + \dots, \quad (47)$$

where the A_i depend only on the coordinates of $\mathbf{R}^{6,1}$, then each term represents a seven-dimensional $U(1)$ gauge field localized on a center of the Taub-NUT space. This is very

appropriate, because there is indeed a $U(1)$ gauge field on each D6-brane: through (47) we are reproducing this known fact from M-theory. Better yet, recall that there are $n(n+1)/2$ holomorphic embeddings of S^2 in a $n+1$ -center Taub-NUT space. An M2-brane wrapped on any of these is some BPS particle, and an anti-holomorphic wrapping is its anti-particle. A closer examination of quantum numbers shows that these wrapped M2-branes carry the right charges and spins to be the non-abelian W -bosons that we know should exist: in the type IIA picture they are the lowest energy modes of strings stretched from one D6-brane to another. It's easy to understand the charge quantum number for the case $n=1$, that is for two D6-branes. The form ω^0 corresponds to what we would call the center-of-mass $U(1)$ of the D6-branes. The form ω^1 is dual to the holomorphic cycle over which we wrap an M2-brane: this is precisely the holomorphic S^2 at $r=\epsilon$ in the Eguchi-Hansen space (11), as discussed after (21). Thus $\int_{S^2} \omega^1 = 1$, which means that an M2-brane on this S^2 does indeed have charge +1 under the $U(1)$ photon which we called A_1 in (47). This is the relative $U(1)$ in the D6-brane description, and the wrapped M2-brane becomes a string stretched between the two D6-branes, which does indeed have charge under the relative $U(1)$.

When two D6-branes come together, one of the holomorphic cycles shrinks to zero size, and there is gauge symmetry enhancement from $U(1)\times U(1)$ to $U(2)$. In the generic situation where the D6-branes are separated, the unbroken gauge group is $U(1)^{n+1}$ on account of the Higgs mechanism. This is a pretty standard aspect of the lore on the relation between M-theory and type IIA, but I find it a particularly satisfying result, because it shows that wrapped M2-branes have to be considered on an equal footing with the degrees of freedom of eleven-dimensional supergravity: in this instance, they conspire to generate $U(n)$ gauge dynamics. See figure 8. Such a conspiracy is one of the reasons why we believe the type duality between IIA string theory and 11-dimensional M-theory goes beyond the supergravity approximation.

Let's return to our earlier observation that any configuration of D6-branes in type IIA must lift to pure geometry in eleven dimensions. Actually, more is true: any type IIA configuration that involves only the metric, the dilaton, and the Ramond-Ramond one-form should lift to pure geometry in eleven dimensions. This means we can include O6-planes as well as D6-branes. Our focus here, however, will be on D6-branes only. Consider, in particular, a set of D6-branes which all share a common $\mathbf{R}^{3,1}$, which we could regard as our own four dimensions. Assume that the configuration preserves $\mathcal{N}=1$ supersymmetry in $d=4$. Then the lift to eleven dimensions should generically be $\mathbf{R}^{3,1}$ times a G_2 holonomy manifold. (We have not entirely ruled out cases where the holonomy group is smaller than G_2 , but we expect such configurations to be quite special, if they exist at all).

Suppose, for instance, that the other six dimensions in the type IIA description are non-compact and asymptotically flat (or else compact/curved on a much larger length

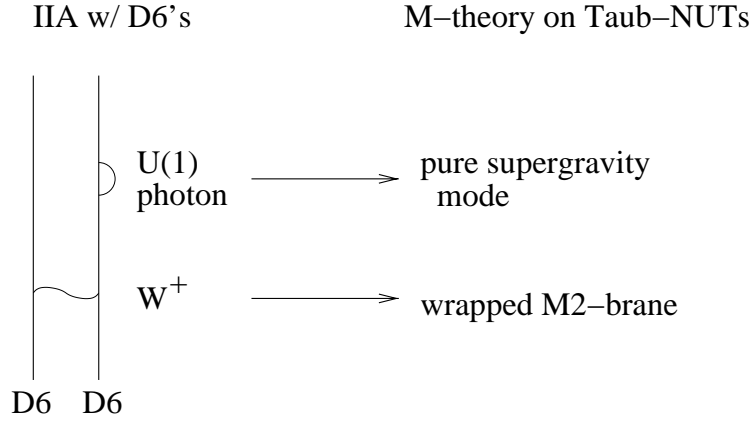


Figure 8: Stretched string degrees of freedom in type IIA lift to very different things in M-theory: a $U(1)$ photon with both its ends on a single D6-brane lifts to a zero-mode of C_{MNP} , whereas a W^+ , with one end on one D-brane and the other on another, lifts to a wrapped M2-brane. These states can be tracked reliably from weak to strong string coupling because they are BPS.

scale than we are considering for now), and that the each D6-brane stretches along some flat $\mathbf{R}^3 \subset \mathbf{R}^6$. When would this configuration preserve $\mathcal{N} = 1$ supersymmetry? The answer to this sort of question was given in one of the early papers on D-branes [?], and it relies on a fermionic Fock-space trick which is also of use in the study of spinors and differential forms on Calabi-Yau spaces. The best way to state the result of [?] is to first choose complex coordinates on \mathbf{R}^6 , call them z^1, z^2 , and z^3 . Obviously there are many inequivalent ways to form the z^i , but suppose we've made up our mind on one for the moment. Now, the $SU(3)$ acting on the z^i by

$$z^i \rightarrow R^i_j z^j \quad \bar{z}^{\bar{i}} \rightarrow R^{\dagger \bar{i}}_{\bar{j}} \bar{z}^{\bar{j}}, \quad (48)$$

is obviously a subgroup of all possible $SO(6)$ rotations. Let us construct Dirac gamma matrices obeying $\{\Gamma^{z^i}, \Gamma^{\bar{z}^{\bar{j}}}\} = 2g^{i\bar{j}} \propto \delta^{i\bar{j}}$. Clearly, the Γ^{z^i} and $\Gamma^{\bar{z}^{\bar{j}}}$ are fermionic lowering/raising operators, up to a normalization. We can define a Fock space vacuum ϵ in the spinor representation of the Clifford algebra via $\Gamma^{z^i} \epsilon = 0$ for all i . The full Dirac spinor representation of $SO(6)$ now decomposes under the inclusion $SU(3) \subset SO(6)$ as $\mathbf{8} = \mathbf{1} \oplus \bar{\mathbf{3}} \oplus \mathbf{3} \oplus \mathbf{1}$. The singlets are ϵ itself and $\Gamma^{\bar{z}^1} \Gamma^{\bar{z}^2} \Gamma^{\bar{z}^3} \epsilon$; the $\bar{\mathbf{3}}$ is $\Gamma^{\bar{z}^{\bar{j}}} \epsilon$; and the $\mathbf{3}$ is $\Gamma^{\bar{z}^{\bar{i}}} \Gamma^{\bar{z}^{\bar{j}}} \epsilon$.

With this mechanism in place, we can state and immediately understand the results of [?]: suppose one D6-brane lies along the \mathbf{R}^3 spanned by the real parts of z^1, z^2 , and z^3 . Consider a collection of other D6-branes on \mathbf{R}^3 's related by various $SU(3)$ rotations, and, optionally, arbitrary translations. The claim is that this configuration preserves $\mathcal{N} = 1$ supersymmetry. To understand why this is so, we need only recall

that the first D6-brane preserves the half of supersymmetry satisfying

$$\tilde{\epsilon}_R = \prod_{i=1}^3 \left(\Gamma^{z^i} + \Gamma^{\bar{z}^{\bar{i}}} \right) \epsilon_L, \quad (49)$$

where $\tilde{\epsilon}_R$ is the right-handed spacetime spinor that comes from the anti-holomorphic sector on the worldsheet, and ϵ_L is the left-handed spacetime spinor that comes from the holomorphic sector. The rotated D6-branes also preserve half of supersymmetry, but a different half, namely

$$\tilde{\epsilon}_R = \prod_{i=1}^3 \left(R^i_j \Gamma^{z^j} + R^{\dagger \bar{i}}_{\bar{j}} \Gamma^{\bar{z}^{\bar{j}}} \right) \epsilon_L. \quad (50)$$

Some supersymmetry is preserved by the total collection of D6-branes iff we can find simultaneous solutions to (49) and (50) for the various $SU(3)$ rotations R^i_j that appear. In fact, if ϵ_L is the Fock space vacuum ϵ tensored with an arbitrary chiral spinor in four-dimensions, and $\tilde{\epsilon}_R$ is $\Gamma^{\bar{z}^1} \Gamma^{\bar{z}^2} \Gamma^{\bar{z}^3} \epsilon$ times the same four-dimensional chiral spinor, then (49) is obviously satisfied; but also (50) is satisfied, because

$$\tilde{\epsilon}_R = \prod_{i=1}^3 R^{\dagger \bar{i}}_{\bar{j}} \Gamma^{\bar{z}^{\bar{j}}} \epsilon_L = (\det R^{\dagger \bar{i}}_{\bar{j}}) \Gamma^{\bar{z}^1} \Gamma^{\bar{z}^2} \Gamma^{\bar{z}^3} \epsilon_L \quad (51)$$

when ϵ_L is as stated above; and $\det R^{\dagger \bar{i}}_{\bar{j}} = 1$ for an $SU(3)$ matrix.

More in fact was shown in [?]: it turns out that $\mathcal{N} = 1$ $d = 4$ chiral matter lives at the intersection of D6-branes oriented at unitary angles in the manner discussed in the previous paragraph. We will not here enter into the discussion in detail, but merely state that the GSO projection that acts on string running from one D6-brane to another one at a unitary angle from the first winds up projecting out the massless fermions of one chirality and leaving the other. Clearly, such strings carry bifundamental charges under the gauge group on the D6-branes they end on; so if one intersects, say, a stack of two coincident D6-branes with another stack of three, the four-dimensional dynamics of the intersection is $U(3) \times U(2)$ gauge theory with chiral “quarks” in the $(\mathbf{3}, \bar{\mathbf{2}})$. This is remarkably similar to the Standard Model! See figure 9. After these lectures were given, work has appeared [?] where explicit compact constructions are given, involving D6-branes and O6-planes, whose low-energy dynamics includes the supersymmetric Standard Model (as well as some other, possibly innocuous, extra degrees of freedom). See also the related work [?]. As is clear from the previous discussion, such constructions lift in M-theory to configurations which involve only the metric, not $G_{(4)}$. Only there are singularities in the eleven-dimensional metric where D6-branes cross. As shown in [?], the singularity in the G_2 holonomy lift of two D6-branes intersecting at unitary angles is precisely the cone over \mathbf{CP}^3 exhibited in (40)

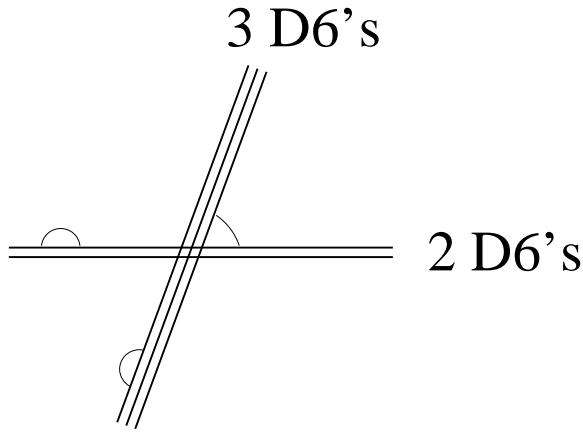


Figure 9: Three coincident D6-branes intersecting two coincident D6-branes at a unitary angle. Strings are shown that give rise to $U(3) \times U(2)$ gauge fields (on the respective D6-brane worldvolumes) and chiral fermions in the $(\mathbf{3}, \bar{\mathbf{2}})$ (at the intersection).

(or, properly, the limit of this geometry where the S^4 shrinks); whereas the singularity in the G_2 holonomy lift of three D6-branes (all at different, unitarily related angles) is the cone over $\frac{SU(3)}{U(1) \times U(1)}$ (of a form very similar to (40), as discussed above). More detail can be found in [?], and also in the earlier work [?], on the D6-brane interpretation of resolving the conical singularity. Oddly, it seems that the generalizations of these G_2 holonomy cones to any number of D6-branes, intersecting all at different angles, is not known. Also, the full geometry interpolating between the near-intersection region, where the metric is nearly conical, and the asymptotic region, where the metric is nearly Taub-NUT close to any single D6-brane, is not known. Finding either sort of generalization of the existing result (40) would be very interesting, and possibly useful for studying the dynamics of M-theory compactifications.

A word of explanation is perhaps in order on why we have focused so exclusively on M-theory geometries with $G_{(4)} = 0$. Really there are two. First, on a compact seven-manifold, there are rather tight constraints on how $G_{(4)}$ may be turned on—see for example [?]. The main context of interest where non-zero $G_{(4)}$ seems necessary is compactifications of Horava-Witten theory: there seems impossible to satisfy the anomaly condition $\text{tr } R \wedge R - \frac{1}{2} \text{tr } F \wedge F = 0$ on each E_8 plane individually, so some $G_{(4)}$ is needed to “soak out” the anomaly. The second reason to consider M-theory geometries first with $G_{(4)} = 0$ is that, in an expansion in the gravitational coupling, the zeroth order equations of motion are indeed Ricci-flatness. For instance, in Horava-Witten compactifications, the necessary $G_{(4)}$ is only a finite number of Dirac units through given four-cycles. As long as only finitely many quanta of $G_{(4)}$ are turned on, and provided the compactification scale is well below the eleven-dimensional Planck scale, one would expect to learn much by starting out ignoring $G_{(4)}$ altogether. This

is not to say that nonzero $G_{(4)}$ won't have some interesting and novel effects: see for example [?, ?]. It is fair to say that our understanding of M-theory compactifications is in a very primitive state, as compared, for instance, to compactifications of type II or heterotic strings. It is to be hoped that this topic will flourish in years to come.

3.1 Addendum: further remarks on intersecting D6-branes

My original TASI lectures ended here, but in view of the continuing interest in G_2 compactifications of M-theory, it seems worthwhile to present a little more detail on intersecting D6-branes and their M-theory lift. This in fact is a subject where rather little is known, so I will in part be speculating about what might be accomplished in further work.

First it's worthwhile to reconsider the work of [?] in light of a particular calibration on \mathbf{R}^6 . Consider the complex three-form $\Omega = dz^1 \wedge dz^2 \wedge dz^3$, where the z^i are, as before, complex coordinates on \mathbf{R}^6 such that the metric takes the standard Kahler form. Ω is called the holomorphic three-form, or the holomorphic volume form, and if space had permitted, some elegant results could have been presented about how the analogous form on a curved complex manifold relates to its complex structure, as well as to covariantly constant spinors, if they exist. Our purpose here is to note that $\text{Re } \Omega$ is a calibration, in the sense explained in section 2.2. Clearly $\text{Re } \Omega$ calibrates the plane in the $\text{Re } z^1$, $\text{Re } z^2$, $\text{Re } z^3$ directions. Any $SU(3)$ change of the coordinates z^i preserves Ω ; in fact such a map is the most general linear map that does so. So it is not hard to convince oneself that all planes related to the one we first mentioned by a $SU(3)$ rotation are also calibrated by $\text{Re } \Omega$. One can now concisely restate the result that D6-branes stretched on $\mathbf{R}^{3,1}$ must be at unitary angles in the remaining \mathbf{R}^6 to preserve supersymmetry: supersymmetric intersecting D6-branes must all be calibrated by $\text{Re } \Omega$, for some suitable choice of the z^i . Choice of the z^i here includes the ability to rotate $\text{Re } \Omega$ by a phase. A more general truth is that supersymmetric D6-branes on a three-cycle of a Calabi-Yau manifold are those calibrated by $\text{Re } \Omega$. Such three-cycles are called special lagrangian manifolds.⁶

It can be shown (see for example [?]) that the G_2 cones over \mathbf{CP}^3 and $\frac{SU(3)}{U(1) \times U(1)}$ are limits of the M-theory lift of two and three D6-branes intersecting at a common point and at unitary angles. It is even known how the G_2 resolutions of these cones corresponds to deformed world-volumes of the D6-branes: for instance, for the cone over \mathbf{CP}^3 , the D6-brane world-volume has an hour-glass shape which is topologically $S^2 \times \mathbf{R}^2$.

⁶We have not been quite precise in the main text: more accurately, D6-branes should wrap special lagrangian manifolds, and such a manifold has the properties that both the Kahler form and $\text{Im } \Omega$ pull back to zero, as well as being calibrated by $\text{Re } \Omega$.

We can describe in a straightforward fashion, though without mathematical rigor, how supersymmetric D6-brane configurations spanning $\mathbf{R}^{3,1}$ and a special lagrangian manifold in \mathbf{R}^6 can be lifted to manifolds of G_2 holonomy in eleven dimensions. As explained in section 3, the eleven-dimensional lift of a single flat D6-brane is Taub-NUT space times $\mathbf{R}^{6,1}$, which we will conveniently write as $\mathbf{R}^{3,1} \times \mathbf{R}^3$. Intuitively speaking, we should be able to lift any D6-brane configuration with no coincident or intersecting D6-branes, just by making an affine approximation to the curving D6-brane world volume at each point, and lifting to Taub-NUT times the D6-brane world-volume times $\mathbf{R}^{3,1}$. From now on let's ignore the $\mathbf{R}^{3,1}$ part. Then in the seven remaining dimensions, the geometry far from any brane is $\mathbf{R}^6 \times S^1$. Near the D6-brane world volume, we cut out a region in \mathbf{R}^6 that surrounds the brane, and since locally this region is $\mathbf{R}^3 \times B^3$, we can replace $\mathbf{R}^3 \times B^3 \times S^1$ in the seven dimensional geometry by \mathbf{R}^3 times a cut-off Taub-NUT. Gluing in the Taub-NUT space should cause only very small discontinuities, which hopefully could be erased through some real analysis.

There is a meaningful point to check, though: in our putative almost- G_2 manifold, formed by gluing into $\mathbf{R}^6 \times S^1$ a cut-off Taub-NUT snaking along what was the D6-brane world volume, we'd like to see that the holonomy on different parts of the "snake" is always (nearly) contained in the same G_2 . To this end, we need to write down a covariantly constant three-form φ for \mathbf{R}^3 times Taub-NUT. This can be done in different ways, because there are different embeddings of $SU(2)$ into G_2 . Let x^1, x^2 , and x^3 be coordinates on \mathbf{R}^3 , and let y^1, y^2, y^3 , and x^{11} be coordinates on Taub-NUT. One choice suggested by the discussion in section 3 is to use the fact that Taub-NUT is hyperkahler, and construct

$$\varphi = \omega_1 \wedge dx^1 + \omega_2 \wedge dx^2 + \omega_3 \wedge dx^3 + dx^1 \wedge dx^2 \wedge dx^3, \quad (52)$$

where the ω_i are the Kahler structures. This is *not* the choice of φ that we will be particularly interested in. Instead, we want a φ which will have some transparent connection with the complex structure of \mathbf{R}^6 . If we choose complex coordinates $z^j = x^j + iy^j$ for $j = 1, 2, 3$, then a D6-brane in the x^1 - x^2 - x^3 plane (or any $SU(3)$ image of it) is calibrated by $\text{Re } \Omega$. The standard Kahler form on $\mathbf{R}^6 = \mathbf{C}^3$ is

$$J = \frac{i}{2} (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3), \quad (53)$$

and one can readily verify that

$$\varphi_0 = \text{Re } \Omega - J \wedge dx^{11} \quad (54)$$

is a G_2 -structure on $\mathbf{R}^6 \times S^1$ (whose holonomy is certainly a subgroup in G_2). Actually, much more is true: the φ_0 in (54) can be constructed in the same way for any $CY_3 \times S^1$, and it represents an inclusion of $SU(3)$ in G_2 . In fact the Joyce construction we

explained in section 3 is believed to generalize to \mathbf{Z}_2 orbifolds of $CY_3 \times S^1$, acting with two fixed points on the S^1 and as an anti-holomorphic involution on the CY_3 .

The obvious vielbein for $X = \mathbf{R}^3 \times (\text{Taub-NUT})$ is

$$\begin{aligned} e^1 &= dx^1 & e^2 &= dx^2 & e^3 &= dx^3 \\ e^4 &= \sqrt{1+H}dy^1 & e^5 &= \sqrt{1+H}dy^2 & e^6 &= \sqrt{1+H}dy^3 \\ e^7 &= \frac{1}{\sqrt{1+H}}(dx^{11} + V), \end{aligned} \tag{55}$$

where

$$dV = *_y dH \quad \text{and} \quad H = \frac{R}{2|\vec{y}|}. \tag{56}$$

Here $*_y$ represents the Hodge dual in the y^j directions only, and $x^{11} \sim x^{11} + 2\pi R$. One can now modify φ_0 slightly to give a G_2 -structure on X :

$$\begin{aligned} \varphi &= dx^1 \wedge dx^2 \wedge dx^3 - (1+H) \left(dx^1 \wedge dy^2 \wedge dy^3 + dy^1 \wedge dx^2 \wedge dy^3 + dy^1 \wedge dy^2 \wedge dx^3 \right) \\ &\quad - J \wedge (dx^{11} + V) \\ &= \varphi_0 - H \left(dx^1 \wedge dy^2 \wedge dy^3 + dy^1 \wedge dx^2 \wedge dy^3 + dy^1 \wedge dy^2 \wedge dx^3 \right) - J \wedge V. \end{aligned} \tag{57}$$

Now, $\varphi \rightarrow \varphi_0$ as $|\vec{y}| \rightarrow \infty$. The key point is that φ_0 is invariant under $SU(3)$ changes of the coordinates z^i : this is so because both Ω and J are $SU(3)$ singlets. So the φ we would construct locally at any point along the lift of the D6-brane world-volume asymptotes to the *same* φ_0 . This is the desired verification that the holonomy of the entire approximation to the seven-dimensional manifold is (nearly) in the same G_2 . I have included the “(nearly)” because of the errors in the affine approximation to the D6-brane world-volume. This error can be uniformly controlled if there are no coincident or intersecting D6-branes. A way to think about it is that we make R much smaller than the closest approach of one part of the world-volume to another.

A remarkable fact is that the deformation of the G_2 -structure, $\varphi - \varphi_0$, in (57), is *linear* in H . This is true despite the fact that the vielbein and the metric are complicated non-linear functions of H . It is tempting to conjecture that an appropriate linear modification of φ_0 , along the lines of (57), will be an *exact* G_2 structure on the whole seven-manifold, even in cases where D6-branes intersect (of course, in such a case one must exclude the singularity right at the intersection). However, we have been unable to verify this.⁷ Knowing the covariantly constant three-form suffices to determine the metric, via the explicit formula

$$\begin{aligned} g_{\mu\nu} &= (\det s_{\mu\nu})^{-1/9} s_{\mu\nu} \quad \text{where} \\ s_{\mu\nu} &= \frac{1}{144} \varphi_{\mu\lambda_1\lambda_2} \varphi_{\nu\lambda_3\lambda_4} \varphi_{\lambda_5\lambda_6\lambda_7} \epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6\lambda_7}, \end{aligned} \tag{58}$$

⁷We thank I. Mitra, O. Evnin, and A. Brandhuber for discussions on this point.

where $\epsilon^{1234567} = \pm 1$, and the sign is chosen to make $s_{\mu\nu}$ positive definite. Convincing oneself of the truth of this formula (which appeared quite early, see for instance [?]) is pretty straightforward: $s_{\mu\nu}$ is symmetric, but scales the wrong way under rigid rescalings of the manifold to be a metric. The determinant factor in the definition of $g_{\mu\nu}$ corrects this problem. The strategy of finding φ first and then deducing $g_{\mu\nu}$ has been of use in a recent investigation in the string theory literature [?].

Acknowledgments

I thank A. Brandhuber, J. Gomis, S. Gukov, and E. Witten for extensive discussions about G_2 holonomy. I would like to express my special appreciation to K. T. Mahanthappa for his many years of dedication to TASI, and to Kathy Oliver for her help in making the school function. This work was supported in part by the DOE under grant DE-FG03-92ER40701.